17/1/22 B.Sc. Part II (Hons.) 4th Paper

Exact diff. Equations (contd.)

$\boxed{\text{Method 4}}$ If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x only i.e. $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ then

$$IF = e^{\int f(x)\,dx}.$$

Example Solve $(x^2+y^2+x)dx + xy\,dy = 0$

Soln Here, $M = x^2+y^2+x$, $N = xy$

as the equation is of the form $Mdx + Ndy = 0$

$\therefore \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y \Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = \frac{1}{x} = f(x) \text{ only}$$

$$\Rightarrow IF = e^{\int f(x)\,dx} = e^{\int \frac{dx}{x}} = e^{\log x} = x.$$

Multiplying the given eqn by IF, we have

$$x(x^2+y^2+x)\,dx + x^2y\,dy = 0$$

$$\Rightarrow x^3dx + x^2dx + (xy^2dx + x^2y\,dy) = 0$$

$$\Rightarrow 2(x^3dx + x^2dx) + (2xy^2dx + 2x^2y\,dy) = 0$$

$$\Rightarrow 2(x^3dx + x^2dx) + d(x^2y^2) = 0$$ Integrating, we get

$$\Rightarrow 2.\frac{x^4}{4} + 2\frac{x^3}{3} + x^2y^2 = \frac{K}{6}$$

$$\Rightarrow 3x^4 + 4x^3 + 6x^2y^2 = K$$

Q Solve $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)dy = 0$.

Soln

Here, $M = y + \frac{y^3}{3} + \frac{x^2}{2}$, $N = \frac{1}{4}(x + xy^2)$

$\Rightarrow \frac{\partial M}{\partial y} = 1 + y^2$, $\frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{3}{4}(1 + y^2)$

$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x\cancel{(1+y^2)}} \times \frac{3}{4}\cancel{(1+y^2)} = \frac{3}{x}$

which is a function of x only, i.e. $f(x)$

$\therefore IF = e^{\int f(x)\,dx} = e^{3\int \frac{dx}{x}} = e^{3\log x} = x^3$.

Multiplying the given equation by IF, we have

$\underbrace{x^3\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)}_{M_1} dx + \underbrace{\frac{x^3}{4}(x + xy^2)}_{N_1} dy = 0$

which is an exact equation as now

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$.

$\therefore$ Its solution is given by

$\int M_1\,dx$ (y as constant) + $\int$(those terms of N_1 free from x) $dy = c$

$\Rightarrow y\int x^3 dx + \frac{y^3}{3}\int x^3 dx + \frac{1}{2}\int x^5 dx + 0 = K$

$\Rightarrow y \cdot \frac{x^4}{4} + \frac{y^3}{3} \cdot \frac{x^4}{4} + \frac{1}{2} \cdot \frac{x^6}{6} = K$

$\Rightarrow x^4\left(\frac{y}{4} + \frac{y^3}{12} + \frac{x^2}{12}\right) = K$